

SKYRMIONS, INSTANTONS, MASS AND CURVATURE

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Abstract

We show that Skyrmons with massless pions in hyperbolic space provide a good approximation to Skyrmons with massive pions in Euclidean space, for a particular relationship between the pion mass and the curvature of hyperbolic space. Using this result we describe how a Skyrmon with massive pions in Euclidean space can be approximated by the holonomy along circles of a Yang-Mills instanton. This is a generalization of the approximation of Skyrmons by the holonomy along lines of an instanton, which is only applicable to massless pions.

1 Introduction

The Skyrme model [17] is a nonlinear theory of pions whose topological soliton solutions are candidates for an effective description of nuclei, with an identification between soliton and baryon numbers. A term can be included in the Skyrme Lagrangian which gives the pions a mass, but as the pion mass is relatively small this term is often neglected. For massless pions a Skyrmion has an algebraic asymptotic behaviour, but if the pions are given a mass then the main effect is that the Skyrmion becomes exponentially localized. Provided the baryon number is low then a small pion mass has little qualitative effect beyond changing the localization. However, a recent study [8] has shown that massive pions can have dramatic effects, if either the baryon number or the pion mass is not small. For massless pions all the known minimal energy Skyrmions have a shell-like structure [7] but for massive pions some of these configurations are no longer bound states, and there appear to be minimal energy Skyrmions which are finite chunks of the Skyrme crystal; this is an infinite triply periodic Skyrme field with a very low energy per baryon [9, 12]. These results motivate a further study of Skyrmions with massive pions.

In this letter we show that there is a surprising similarity between Skyrmions with massive pions in Euclidean space and Skyrmions with massless pions in hyperbolic space, with a relation between the pion mass and the curvature of hyperbolic space. We map hyperbolic Skyrmions to Euclidean Skyrmions, by identifying the hyperbolic and Euclidean radii, and find that this gives very good approximations for a range of pion masses.

There are some very useful approximate methods for studying Skyrmions with massless pions. One approach is the rational map ansatz [10], where the angular dependence of the Skyrmion is determined by a rational map between Riemann spheres. This approach is easily extended to the case of massive pions, where again it provides good results [8], but it can only describe Skyrmions which are shell-like. For massive pions some minimal energy Skyrmions are not shell-like so an alternative approach is required.

Skyrmions with massless pions can be approximated by computing the holonomy of $SU(2)$ Yang-Mills instantons in Euclidean \mathbb{R}^4 along lines parallel to the Euclidean time axis [5]. It seems likely that all minimal energy Skyrmions can be approximated by the holonomy of a suitable instanton and a number of examples have been studied in detail [6, 13, 18, 19]. This approach is more general than the rational map ansatz, since it is not restricted to shell-like configurations. Furthermore, it is known [15] that there is an instanton on \mathbb{T}^4 whose holonomy gives an approximation to the Skyrme crystal, so it seems likely that crystal chunks could also be approximated by instanton holonomies.

Instanton generated Skyrme fields have the correct algebraic decay for Skyrmions with massless pions, but they can not be used to approximate Skyrmions with massive pions, since the decay is too slow to yield finite energy. It would therefore be desirable to have a generalization of the instanton holonomy method that could produce Skyrme fields with the exponential decay appropriate for massive pions. Such a generalization is presented here. Computing the holonomy of an instanton along particular circles in \mathbb{R}^4 yields a Skyrme field in hyperbolic 3-space, and applying our mapping to this hyperbolic Skyrme field produces a Skyrme field in Euclidean space with exponential decay. In the limit of

massless pions this procedure coincides with the usual instanton holonomy along lines, but it is applicable to any value of the pion mass. We apply this scheme explicitly to the single Skymion for a range of pion masses and find that the approximation actually improves with increasing pion mass, so that the error is always less than the usual case with massless pions.

2 Hyperbolic Skymions and the pion mass

The static energy of the $SU(2)$ -valued Skyrme field $U(\mathbf{x})$ on a 3-dimensional Riemannian manifold M with metric $ds^2 = g_{ij}dx^i dx^j$ is given by

$$E = \frac{1}{12\pi^2} \int \left\{ -\frac{1}{2} \text{Tr}(R_i R^i) - \frac{1}{16} \text{Tr}([R_i, R_j][R^i, R^j]) + m^2 \text{Tr}(1 - U) \right\} \sqrt{g} d^3x, \quad (2.1)$$

where $R_i = (\partial_i U)U^\dagger$ is the $su(2)$ -valued current, g denotes the determinant of the metric and m is the pion mass parameter.

It is easy to see that m is the (tree-level) mass of the pions by making contact with the nonlinear pion theory via $U = \sigma + i\boldsymbol{\pi} \cdot \boldsymbol{\tau}$, where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the triplet of pion fields, $\boldsymbol{\tau}$ denotes the triplet of Pauli matrices and σ is determined by the constraint $\sigma^2 + \boldsymbol{\pi} \cdot \boldsymbol{\pi} = 1$.

In this paper we shall be concerned with two choices for M . The first choice is simply Euclidean space $M = \mathbb{R}^3$, where we are mainly concerned with the properties of Skymions for varying values of the pion mass m . The Skyrme model has energy and length units which have been scaled out in the expression (2.1). These energy and length units must be fixed by comparison with experiment, and then m denotes the pion mass in these units. The standard approach to fixing these units involves fitting the masses of the proton and delta resonance [2] and this leads to a value [1] of $m = 0.526$. However, other approaches to fixing the units are possible and these would produce different values of m corresponding to the physical pion mass, so it is worth investigating the properties of Skymions as a function of m .

The second choice we shall discuss is where $M = \mathbb{H}_\kappa^3$, hyperbolic 3-space of constant negative curvature $-\kappa^2$. In this case we restrict to massless pions ($m = 0$) and consider the properties of Skymions for varying values of the curvature $-\kappa^2$.

In the following we shall make a surprising connection between these two apparently quite different situations.

Let us first consider the Euclidean case. The single Skymion has the hedgehog form

$$U = \exp(if(r)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}) \quad (2.2)$$

where $f(r)$ is the radial profile function with boundary conditions $f(0) = 2\pi$ and $f(\infty) = 0$. The energy of this field is given by

$$E = \frac{1}{3\pi} \int \left(r^2 f'^2 + 2(f'^2 + 1) \sin^2 f + \frac{\sin^4 f}{r^2} + 2m^2 r^2 (1 - \cos f) \right) dr. \quad (2.3)$$

The energy minimizing profile function has the asymptotic exponential decay

$$f \sim \frac{A}{r} e^{-mr} \quad (2.4)$$

but in the massless pion limit ($m = 0$) this is replaced by the algebraic form $f \sim Ar^{-2}$.

A numerical solution of the ordinary differential equation which follows from (2.3) allows the Skyrmin energy to be computed as a function of the pion mass and this is displayed as the solid curve in Fig. 1. The profile functions for $m = 0, 1, 2, 3$ are shown as the solid

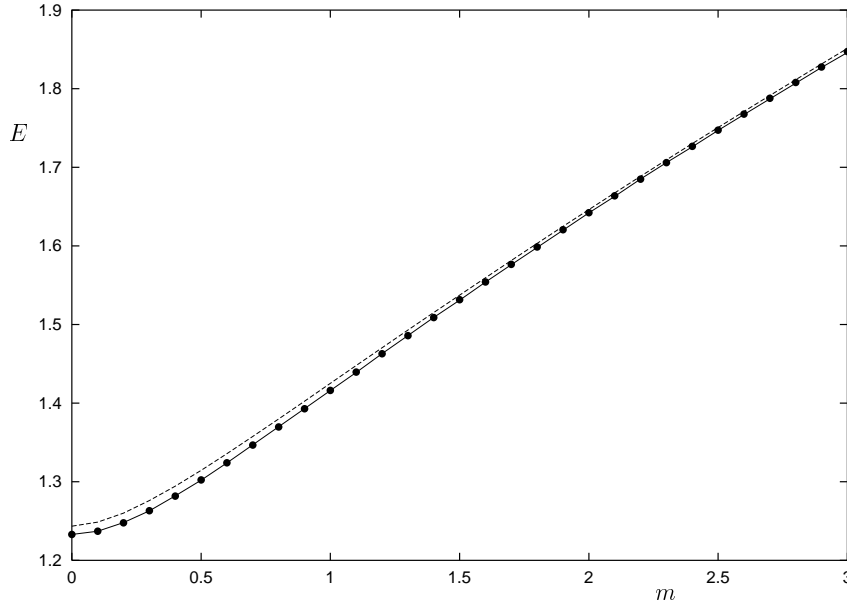


Figure 1: The Skyrmin energy as a function of the pion mass; exact result (solid curve), hyperbolic approximation (circles), instanton approximation (dashed curve).

curves in Fig. 2, demonstrating the increased localization with increasing pion mass.

Now consider hyperbolic Skyrmons with massless pions. In spherical coordinates the metric of \mathbb{H}_κ^3 takes the form

$$ds^2(\mathbb{H}_\kappa^3) = d\rho^2 + \frac{\sinh^2 \kappa \rho}{\kappa^2} (d\theta^2 + \sin^2 \theta d\phi^2) \quad (2.5)$$

where ρ is the hyperbolic radius and the curvature of hyperbolic space is $-\kappa^2$. Note that in the limit as $\kappa \rightarrow 0$ the Euclidean metric is recovered with $\rho = r$ the usual radial coordinate.

The hedgehog form

$$U = \exp(iF(\rho)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}) \quad (2.6)$$

gives the energy of the hyperbolic Skyrmin to be

$$E = \frac{1}{3\pi} \int \left(F'^2 \frac{\sinh^2 \kappa \rho}{\kappa^2} + 2(F'^2 + 1) \sin^2 F + \frac{\kappa^2 \sin^4 F}{\sinh^2 \kappa \rho} \right) d\rho. \quad (2.7)$$

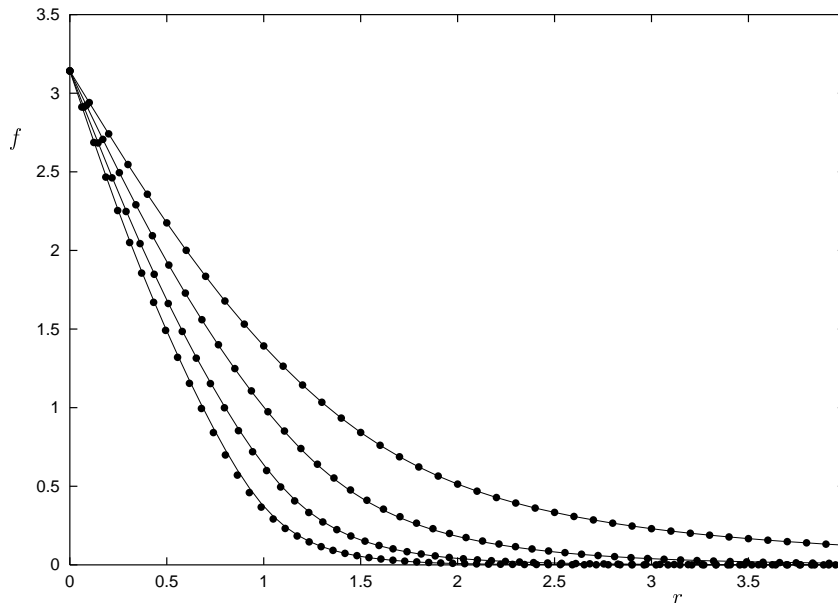


Figure 2: Profile functions for pion masses $m = 0, 1, 2, 3$; exact result (solid curves) and hyperbolic approximation (circles). The curves are more localized for larger m .

The energy minimizing profile function again has an asymptotic exponential decay

$$F \sim Ae^{-2\kappa\rho}. \quad (2.8)$$

Although the Skyrme energy functions in Euclidean space with massive pions and hyperbolic space with massless pions are quite different they coincide in the limits $m = 0$ and $\kappa \rightarrow 0$, where the massless pion Euclidean Skyrmion is recovered. Both modifications away from this limit yield exponentially localized solutions. Furthermore, close to the origin of hyperbolic space it increasingly resembles Euclidean space so the influence of both curvature and the pion mass should be greatest outside the Skyrmons core. Thus it seems reasonable to ask if there are any similarities between these two systems.

To compare a Skyrme field in hyperbolic space to one in Euclidean space we simply map the hyperbolic radius ρ to the Euclidean radius r (note that this requires a choice of origin and so breaks the translation invariance of \mathbb{R}^3). In the case of a single Skyrmion this means comparing the profile functions $f(r)$ and $F(\rho = r)$. It is interesting to see if there is a relationship between the two deformation parameters m and κ so that these profile functions are similar.

Comparing the leading order of the asymptotic decays (2.4) and (2.8) suggests the simple linear relationship $\kappa = m/2$. However, since this result is obtained using only the large radius behaviour, where the difference between Euclidean and hyperbolic space is greatest, then we expect this simple linear relationship to be only a rough guide. As the size of a Skyrmion depends both on the pion mass and the curvature of hyperbolic space

then a better method for determining a possible relationship between κ and m is to fix the size of the Skyrmion to be the same in both cases. The size of a Skyrmion is defined to be the radius at which the profile function takes the value $\pi/2$, so that the σ field vanishes. Thus for each value of m we compute r_* such that $f(r_*) = \pi/2$ and then determine κ by the requirement that $F(r_*) = \pi/2$. The result is the relationship $\kappa(m)$ plotted in Fig. 3, which is reasonably close to the naive linear estimate $\kappa = m/2$.

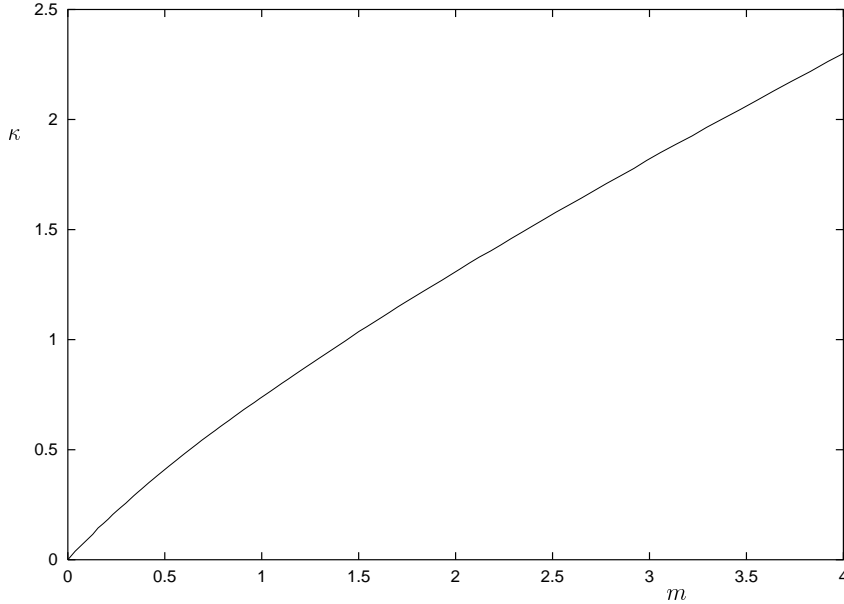


Figure 3: κ as a function of m , obtained by equating the size of a Skyrmion.

Given the relationship $\kappa(m)$ we can now investigate whether the Skyrmion in hyperbolic space of curvature $-(\kappa(m))^2$ with massless pions is a good approximation to the Skyrmion in Euclidean space with pion mass m . To do this we take the hyperbolic Skyrmion, identify hyperbolic and Euclidean radii by making the replacement $\rho \mapsto r$, and compute the Euclidean energy of this Skyrme field, to determine its excess over the exact energy. The energy of the hyperbolic approximation is displayed as the circles in Fig. 1 for a range of m . It can be seen from this figure that the hyperbolic approximation is remarkably close to the true solution, and in fact the excess energy is always less than 0.1% over this range of pion masses. In Fig. 2 the circles display the hyperbolic profile function $F(\rho = r)$ for $\kappa(m)$ with $m = 0, 1, 2, 3$, for comparison with the true profile functions $f(r)$ (solid curves). Again this demonstrates that the hyperbolic approximation is remarkably accurate.

So far our study has been limited to the single Skyrmion, but now we turn to multi-Skyrmions. Ideally, the comparison of hyperbolic and Euclidean multi-Skyrmions would involve full numerical simulations of both nonlinear field theories, but this is computationally expensive. As an alternative we shall investigate multi-Skyrmions within the framework of the rational map ansatz. For Euclidean Skyrmons with massive pions the rational

map approximation has already been compared to full field simulations [8] and found to be an excellent approximation for shell-like configurations for the whole range of pion masses studied. At least for baryon numbers $B \leq 4$ it appears that shell-like Skyrmions survive as the minimal energy configurations even for quite large pion masses, so the rational map approach is justified.

We can treat both Euclidean and hyperbolic Skyrmions simultaneously by first considering the Skyrme model on hyperbolic space with a pion mass \widetilde{m} and then taking the limit $\kappa \rightarrow 0$ with $\widetilde{m} = m$ to recover the Euclidean case or the limit $\widetilde{m} = 0$ with $\kappa = \kappa(m)$ to obtain the hyperbolic massless pion case.

The rational map ansatz [10] constructs a Skyrme field with baryon number B from a degree B rational map between Riemann spheres. Although this ansatz does not give exact multi-Skyrmion solutions of the static Skyrme equations, it produces approximations which (at least in the Euclidean case) have energies only a few percent above the numerically computed solutions. Furthermore, the approximation of the angular dependence involved in this approach should produce similar errors in both Euclidean and hyperbolic space, so we can use the rational map determined energies to compare these two cases without worrying about the small common error.

To present the rational map ansatz we first introduce the Riemann sphere coordinate $z = e^{i\phi} \tan(\theta/2)$, and let $R(z)$ be a degree B rational map between Riemann spheres, that is, $R = p/q$ where p and q are polynomials in z such that $\max[\deg(p), \deg(q)] = B$, and p and q have no common factors. Given such a rational map the ansatz for the Skyrme field in hyperbolic space is

$$U(\rho, z) = \exp \left[\frac{iF(\rho)}{1 + |R|^2} \begin{pmatrix} 1 - |R|^2 & 2\bar{R} \\ 2R & |R|^2 - 1 \end{pmatrix} \right], \quad (2.9)$$

where $F(\rho)$ is a real profile function satisfying the boundary conditions $F(0) = \pi$ and $F(\infty) = 0$. This profile function is determined by minimization of the energy of the field (2.9) given a particular rational map R .

Substitution of the ansatz (2.9) into the hyperbolic Skyrme energy results in the following expression

$$E = \frac{1}{3\pi} \int \left(F'^2 \frac{\sinh^2 \kappa \rho}{\kappa^2} + 2B(F'^2 + 1) \sin^2 F + \mathcal{I} \frac{\kappa^2 \sin^4 F}{\sinh^2 \kappa \rho} + 2\widetilde{m}^2 \frac{\sinh^2 \kappa \rho}{\kappa^2} (1 - \cos F) \right) d\rho \quad (2.10)$$

where \mathcal{I} denotes the integral

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i \, dz d\bar{z}}{(1 + |z|^2)^2}. \quad (2.11)$$

For $B = 2, 3, 4$ the rational maps which minimize \mathcal{I} are given by

$$R = z^2, \quad R = \frac{z^3 - \sqrt{3}iz}{\sqrt{3}iz^2 - 1}, \quad R = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}. \quad (2.12)$$

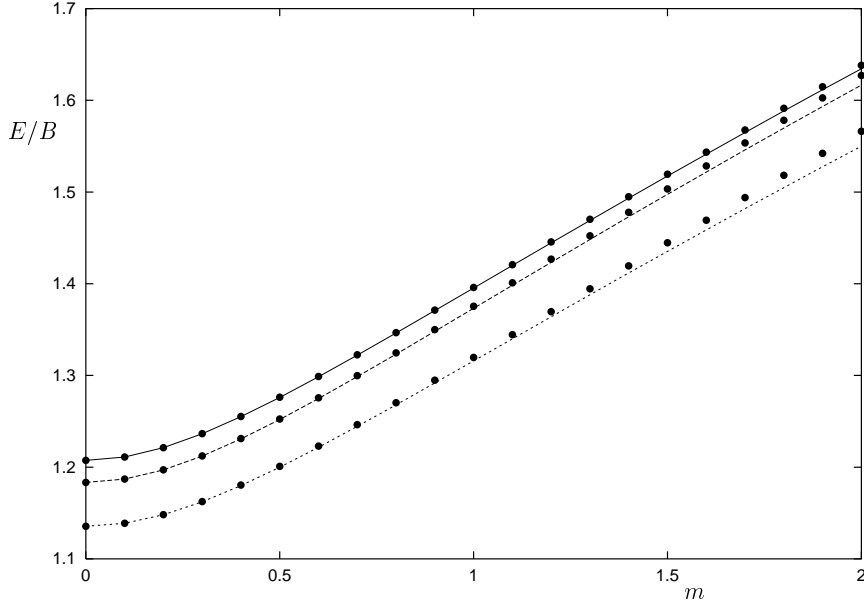


Figure 4: The energy per baryon E/B as a function of the pion mass m for $B = 2$ (solid curve), $B = 3$ (dashed curve), $B = 4$ (dotted curve). The circles denote the energies of the hyperbolic approximations.

Using these maps we first compute the Euclidean energy ($\kappa \rightarrow 0$) with pion mass $\tilde{m} = m$. The results are presented in Fig. 4 where we plot the energy per baryon E/B for $B = 2$ (solid curve), $B = 3$ (dashed curve), $B = 4$ (dotted curve). Next we use the same rational maps and compute the hyperbolic Skyrme fields for massless pions ($\tilde{m} = 0$) with $\kappa = \kappa(m)$, and calculate their Euclidean energy with pion mass m . The energies of these hyperbolic Skyrme approximations are shown as the circles in Fig. 4. We see from this figure that the hyperbolic Skyrme fields are also a good approximation to multi-Skyrmions, though for increasing pion mass the errors are greater for the larger baryon numbers. Note that at a pion mass around the standard one used in the literature, $m = 0.526$, the errors are very small even for the larger baryon numbers.

In this section we have demonstrated that hyperbolic Skyrmons with massless pions provide a good approximation to Euclidean Skyrmons with massive pions. However, hyperbolic Skyrmons are at least as difficult to compute as Euclidean Skyrmons, so it might appear that this connection is not very useful. There are two main reasons why the relation with hyperbolic Skyrmons is of interest.

The first reason is that some insight into the more complicated effects of the pion mass, such as the emergence of minimal energy crystal chunks, might be found by studying the geometry of hyperbolic space. For example, the infinite curvature limit may yield some simplifications as it does in the study of hyperbolic monopoles [4]. Also, the extra term in

the Skyrme Lagrangian that gives the pions a mass is only really determined upto quadratic order in the pion fields. The term that is generally used is just the simplest possibility, and therefore the most natural from one point of view, but the correspondence with hyperbolic Skymions may suggest an alternative choice motivated by the geometry.

The second reason is a more practical one, in that it will allow us to approximate Euclidean Skymions with massive pions in terms of the holonomy of Yang-Mills instantons. This is explained in the following section.

3 Instanton holonomies

Let us first recall how Euclidean Skymions with massless pions can be approximated using Yang-Mills instantons. This scheme was introduced in [5] and involves computing the holonomy of $SU(2)$ instantons in Euclidean \mathbb{R}^4 along lines parallel to the x_4 -axis. Explicitly, the prescription for the Skyrme field is to take

$$U(\mathbf{x}) = \mathcal{P} \exp \left(\int_{-\infty}^{\infty} A_4(\mathbf{x}, x_4) dx_4 \right) \quad (3.1)$$

where \mathcal{P} denotes path ordering and A_μ is the gauge potential of a Yang-Mills instanton in \mathbb{R}^4 , and where $x = (\mathbf{x}, x_4) = (x_1, x_2, x_3, x_4)$. This holonomy is really along a closed loop in S^4 , and is almost gauge invariant. The only effect of a gauge transformation $g(x)$ is to conjugate $U(\mathbf{x})$ by a fixed element $g(\infty)$, and this simply corresponds to an isospin rotation of the Skyrme field.

If A_μ is the field of a charge N instanton then it follows from general topological considerations that the resulting Skyrme field has baryon number $B = N$. The construction yields an $(8N - 1)$ -dimensional family of Skyrme fields from the $8N$ -dimensional moduli space of charge N instantons, and although such fields are never exact solutions of the Skyrme equation, some are good approximations to minimal energy Skymions and other important field configurations [6, 13, 18, 19].

As the simplest example, consider the charge one instanton centred at the origin with width λ . This is given by the 't Hooft ansatz

$$A_\mu = \frac{i}{2} \sigma_{\mu\nu} \partial_\nu \log \zeta \quad (3.2)$$

where

$$\zeta = 1 + \frac{\lambda^2}{|x|^2} \quad (3.3)$$

and $\sigma_{\mu\nu}$ is the anti-symmetric anti-self-dual tensor defined by $\sigma_{i4} = \tau_i$, $\sigma_{ij} = \varepsilon_{ijk} \tau_k$.

The holonomy of this instanton generates a Skyrme field of the hedgehog form with a profile function given by

$$f(r) = \pi \left[1 - \left(1 + \frac{\lambda^2}{r^2} \right)^{-1/2} \right]. \quad (3.4)$$

Instantons are scale invariant, so the parameter λ is arbitrary and can be chosen to minimize the energy of the resulting Skyrme field. The appropriate value of the scale is $\lambda^2 = 2.11$, and then the energy is $E = 1.243$, which is only 1% above that of the true Skyrme solution.

The instanton approximation has been very useful for studying massless pion Skyrme fields, so it would be useful if a similar approach was available in the massive pion case. However, note that the asymptotic decay of the instanton generated profile function is $f \sim \lambda^2/(2r^2)$ and this has the correct form for massless pions, but for massive pions this algebraic decay produces infinite energy. Thus this form of the instanton holonomy method is not applicable to massive pions.

Skyrmions on hyperbolic space can be approximated by instanton holonomies along circles in \mathbb{R}^4 [14]. We can combine this approach with the results of the previous section, relating hyperbolic Skyrme fields to Euclidean Skyrme fields with massive pions, to obtain a generalization of the instanton approximation that is valid for all pion masses. The details are as follows.

The starting point is to observe that there is a conformal equivalence between $\mathbb{R}^4 - \mathbb{R}^2$ and $\mathbb{H}_\kappa^3 \times S_{\kappa^{-1}}^1$, where $S_{\kappa^{-1}}^1$ denotes the circle of radius κ^{-1} . One way to see this explicitly is to introduce toroidal coordinates $(\rho, \theta, \phi, \chi)$ on \mathbb{R}^4 given by

$$x = \frac{1}{\cosh(\kappa\rho) + \cos\chi} (\sinh(\kappa\rho) \sin\theta \cos\phi, \sinh(\kappa\rho) \sin\theta \sin\phi, \sinh(\kappa\rho) \cos\theta, \sin\chi) \quad (3.5)$$

where κ is a real parameter. It is then easy to check that the metric on \mathbb{R}^4 becomes

$$ds^2(\mathbb{R}^4) = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = \frac{(ds^2(\mathbb{H}_\kappa^3) + \kappa^{-2}d\chi^2)}{(\cosh(\kappa\rho) + \cos\chi)^2} \quad (3.6)$$

where $ds^2(\mathbb{H}_\kappa^3)$ is the metric on hyperbolic 3-space with spherical coordinates (ρ, θ, ϕ) as given in equation (2.5). Dropping the conformal factor in the final expression above we obtain hyperbolic 3-space and a circle of radius κ^{-1} with coordinate χ .

To obtain a Skyrme field in hyperbolic space the instanton holonomy is computed along the circles parametrized by χ , that is

$$U(\rho, \theta, \phi) = \mathcal{P} \exp \left(\int_0^{2\pi} A_\chi d\chi \right) \quad (3.7)$$

where A_χ is the component of the gauge potential associated with the coordinate χ .

As mentioned earlier, the construction of Skyrme fields on \mathbb{R}^3 from instanton holonomies along lines is essentially gauge invariant, due to the fact that all the lines go through the point at infinity. However, the circles whose holonomy we use to generate hyperbolic Skyrme fields have no points in common, so the holonomy is only defined up to conjugation unless we fix a gauge on the whole of hyperbolic space. Explicitly, under a gauge transformation $g(\rho, \theta, \phi, \chi)$ the holonomy (3.7) transforms as

$$U(\rho, \theta, \phi) \mapsto g^{-1}(\rho, \theta, \phi, 0) U(\rho, \theta, \phi) g(\rho, \theta, \phi, 0) \quad (3.8)$$

which is a local isospin rotation.

There is a rather natural resolution to this problem, namely we fix the gauge by requiring the radial gauge $A_\rho = 0$. This requires a choice of origin but this symmetry breaking choice is already needed in order to apply our mapping from hyperbolic Skyrmons to Euclidean Skyrmons. The upshot is that the Skyrme fields from instantons construction works with full symmetry for the Euclidean case but only with rotational symmetry (around an origin) for the hyperbolic case. This is quite natural since hyperbolic space has no translations.

Let us consider the simplest case of a single Skyrmion. For the charge one instanton in the 't Hooft gauge (3.2) then A_χ has the form $A_\chi = a_\chi(\rho, \chi)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}$, where $a_\chi(\rho, \chi)$ is independent of θ and ϕ . In this gauge A_ρ is non-zero but it has the form $A_\rho = a_\rho(\rho, \chi)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}$, where again $a_\rho(\rho, \chi)$ is independent of θ and ϕ . Since A_χ and A_ρ both have the same direction in the Lie algebra, which furthermore is independent of both ρ and χ , then the gauge transformation to the radial gauge $A_\rho = 0$ has no effect on the holonomy (3.7) and therefore does not need to be found explicitly.

Computing the charge one instanton holonomy yields a hyperbolic Skyrme field of the hedgehog form (2.6) with profile function [14]

$$F(\rho) = \pi \left(1 - \sqrt{\frac{\sinh^2(\kappa\rho)}{\kappa^2\beta^2 + \sinh^2(\kappa\rho)}} \right) \quad (3.9)$$

where β is related to the instanton scale λ via $\beta = 2\lambda\kappa^{-1}/(1 + \lambda^2)$.

Note that in the zero curvature limit, $\kappa \rightarrow 0$ with $\rho = r$, the profile function (3.9) reverts to the Euclidean form (3.4). The fact that in the zero curvature limit the hyperbolic Skyrme field obtained from the holonomy along circles reverts to the Euclidean Skyrme field constructed from the holonomy along lines is a general feature, and corresponds to the fact that the holonomy is computed along circles of radius κ^{-1} which degenerate to lines.

For non-zero curvature the instanton generated hyperbolic Skyrme field has an exponential decay, as can be seen explicitly in the case of the single Skyrmion profile function (3.9). Applying the mapping from hyperbolic to Euclidean space, by identifying radial coordinates, therefore produces instanton generated Euclidean Skyrme fields which can be used to approximate Skyrmons with massive pions.

Considering the single Skyrmion in detail, we take the instanton generated Skyrme field with profile function (3.9) and make the replacement $\rho \mapsto r$. We then compute the Euclidean energy of this approximation as a function of the pion mass, minimizing over the instanton scale, and with κ determined by the pion mass using the relation displayed in Fig. 3. The result is displayed as the dashed curve in Fig. 1. For zero pion mass the result of ref.[5] is reproduced, with the approximation exceeding the true energy by around 1%, but as the pion mass increases it can be seen from Fig. 1 that the dashed curve approaches the solid curve (the true energy), showing that the approximation actually improves with increasing pion mass, at least for the range of values considered.

As further indication that the instanton approximation improves with increasing pion mass we compare in Fig. 5 the true profile functions (solid curves) and the instanton generated profile functions (dashed curves) for pion masses $m = 0, 1, 2, 3$. The more localized curves correspond to larger values of m . It is evident from this figure that the instanton generated fields approximate the exact solutions with an improved accuracy compared to the massless pion case. The instanton approximation for massless pions is itself quite accurate and has proved useful for studying Skyrmions in a number of ways, so we expect that our modified instanton proposal should be useful in the study of Skyrmions with massive pions.

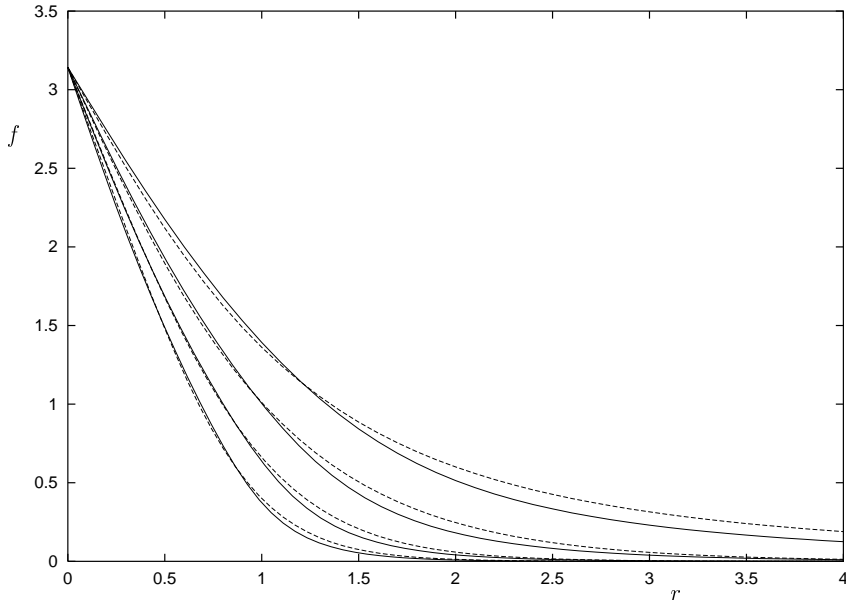


Figure 5: Profile functions for pion masses $m = 0, 1, 2, 3$; exact results (solid curves) and instanton approximation (dashed curves). The curves are more localized for larger m .

4 Conclusion

We have shown that, for a particular relationship between the pion mass and the curvature of hyperbolic space, there is a surprising similarity between Euclidean Skyrmions with massive pions and hyperbolic Skyrmions with massless pions. The latter provide very good approximations to the former and we have shown how this can be used to approximate Euclidean Skyrmions with a pion mass by the holonomy along circles of Yang-Mills instantons.

The connection between Euclidean and hyperbolic Skyrmions suggests that it might be interesting to investigate hyperbolic Skyrmions in more detail. Particularly interesting

aspects include the behaviour of Skyrmions as the curvature tends to infinity, and the interpretation of curvature as a Euclidean pion mass, which may lead to a more geometrically natural mass term.

Given that the instanton holonomy approximation can now be applied to Skyrmions with massive pions it is of interest to determine the instantons which are relevant in this context. In particular, the results of ref.[8] suggest that the structure and symmetries of some minimal energy Skyrmions will be quite different from the massless pion situation, and it might be possible to understand this in terms of instantons. For example, there is evidence that for massive pions the minimal energy charge 32 Skyrmion is a crystal chunk with cubic symmetry. It would therefore be interesting to know if a charge 32 instanton exists whose holonomy is a good approximation to this Skyrmion. One way to approach this problem is to attempt to construct all charge 32 instantons with cubic symmetry and then investigate whether any of these have the correct properties required of a crystal chunk. However, for such a large charge there is likely to be a fairly big moduli space of such symmetric solutions, so it may be necessary to first determine a good method for identifying instantons which resemble crystal chunks.

Crystal chunks are not only interesting for instantons and Skyrmions, but also for monopoles. It is known that there are many similarities between monopoles and Skyrmions (see eg. ref.[16]) and circle invariant instantons can be identified with hyperbolic monopoles [3], so it is natural to wonder whether monopole chunks exist in some form, even though there is no infinite monopole crystal. There is a diffeomorphism, which preserves rotational symmetry, between the moduli space of charge N $SU(2)$ monopoles and the moduli space of degree N rational maps between Riemann spheres [11]. It is fairly easy to construct a two-parameter family of degree 32 rational maps with cubic symmetry, using the Klein polynomials of the tetrahedron, but again it is not easy to see if any member of this family corresponds to a charge 32 monopole which resembles the crystal chunk of the charge 32 Skyrmion. As in the instanton situation it would be very useful if a property of this data could be identified that would distinguish chunk-like monopoles (if they exist) from those which are shell-like.

References

- [1] G. S. Adkins and C. R. Nappi, The Skyrme model with pion masses, *Nucl. Phys.* **B233**, 109 (1984).
- [2] G. S. Adkins, C. R. Nappi and E. Witten, Static properties of nucleons in the Skyrme model, *Nucl. Phys.* **B228**, 552 (1983).
- [3] M. F. Atiyah, Magnetic monopoles in hyperbolic spaces, in *M. Atiyah: Collected Works, vol. 5*, Oxford, Clarendon Press, 1988.
- [4] M. F. Atiyah, Magnetic monopoles and the Yang-Baxter equation, *Int. J. Mod. Phys.* **A6**, 2761 (1991).

- [5] M. F. Atiyah and N. S. Manton, Skyrmons from instantons, *Phys. Lett.* **B222**, 438 (1989).
- [6] M. F. Atiyah and N. S. Manton, Geometry and kinematics of two Skyrmons, *Commun. Math. Phys.* **153**, 391 (1993).
- [7] R. A. Battye and P. M. Sutcliffe, Solitonic fullerene structures in light atomic nuclei, *Phys. Rev. Lett.* **86**, 3989 (2001); Skyrmons, fullerenes and rational maps, *Rev. Math. Phys.* **14**, 29 (2002).
- [8] R. A. Battye and P. M. Sutcliffe, Skyrmons and the pion mass, hep-ph/0410157, to appear in *Nucl. Phys.* **B**.
- [9] L. Castillejo, P. S. J. Jones, A. D. Jackson, J. J. M. Verbaarschot and A. Jackson, Dense Skyrmon systems, *Nucl. Phys.* **A501**, 801 (1989).
- [10] C. J. Houghton, N. S. Manton and P. M. Sutcliffe, Rational maps, monopoles and Skyrmons, *Nucl. Phys.* **B510**, 507 (1998).
- [11] S. Jarvis, A rational map for Euclidean monopoles via radial scattering, *J. reine angew. Math.* **524**, 17 (2000).
- [12] M. Kugler and S. Shtrikman, A new Skyrmon crystal, *Phys. Lett.* **B208**, 491 (1988); Skyrmon crystals and their symmetries, *Phys. Rev.* **D40**, 3421 (1989).
- [13] R. A. Leese and N. S. Manton, Stable instanton-generated Skyrme fields with baryon numbers three and four, *Nucl. Phys.* **A572**, 575 (1994).
- [14] N. S. Manton and T. M. Samols, Skyrmons on S^3 and H^3 from instantons, *J. Phys.* **A23**, 3749 (1990).
- [15] N. S. Manton and P. M. Sutcliffe, Skyrme crystal from a twisted instanton on a four-torus, *Phys. Lett.* **B342**, 196 (1995).
- [16] N. S. Manton and P. M. Sutcliffe, *Topological Solitons*, Cambridge University Press (2004).
- [17] T. H. R. Skyrme, A nonlinear field theory, *Proc. R. Soc. Lond.* **A260**, 127 (1961).
- [18] M. A. Singer and P. M. Sutcliffe, Symmetric instantons and Skyrme fields, *Nonlinearity* **12**, 987 (1999).
- [19] P. M. Sutcliffe, Instantons and the buckyball, *Proc. R. Soc. Lond.* **A460**, 2903 (2004).